

February 1, 2008

Geometric Phases, Coherent States and Resonant Hamiltonians

MASAO MATSUMOTO

1-12-32 Kuzuha Asahi, Hirakata City, Osaka 573, JAPAN

HIROSHI KURATSUJI

*Department of Physics, Ritsumeikan
University-BKC, Kusatsu City 525-77, JAPAN*

ABSTRACT

We study characteristic aspects of the geometric phase which is associated with the generalized coherent states. This is determined by special orbits in the parameter space defining the coherent state, which is obtained as a solution of the variational equation governed by a simple model Hamiltonian called the "resonant Hamiltonian". Three typical coherent states are considered: SU(2), SU(1,1) and Heisenberg-Weyl. A possible experimental detection of the phases is proposed in such a way that the geometric phases can be discriminated from the dynamical phase.

1. Introduction

The main feature of quantum mechanics is the existence of the probability amplitude which underlies all the atomic processes.^[1] In particular, the phase factor of probability amplitude has recently renewed interest, which is inspired by a specific motivation. Namely, if one considers the cyclic change in quantum system, one gets the so-called geometric phases.^[2] From the historical point of view, Dirac was the first who recognized the geometric phase in the form of a non-integrable phase (or path dependent phase factor).^[3] Apart from this monumental work, the development of the geometric phases has, roughly speaking, two aspects. One aspect is that the origin of geometric phases dates back to the Bohr-Sommerfeld factor $\exp[i \oint pdx]$ which is closely connected with the symplectic (canonical) structure of the classical phase space. From the modern point of view, this original form of the geometric phases has been further extended to the one defined over the generalized (or curved) phase space.^[4] The present paper is mainly motivated by this aspect to obtain much deeper understanding of the characteristic features of the geometric phases associated with the generalized phase space. The other aspect is that the geometric phase in its literal sense has been known since early sixties in connection with molecular physics^[5] Then one has been led to the final goal of the nowadays famous quantum adiabatic phase; alias "Berry's phase", which has been formulated by the conventional Schrödinger equation^[6-7] or path integral method.^[8] The contributions in the early development of this topics as well as the historical ones may be found in the reprint book edited by Shapere and Wilczek.^[2]

What we want to address here is to investigate some specific aspect of the geometric phases formulated by path integral in terms of the so-called generalized coherent state, which has originally been studied for different purpose, (see, e.g. the paper^[9]) and later has been restudied for the purpose of describing the geometric phases without recourse to the adiabaticity.^[4] Although the utility of this aspect has not yet been widely appreciated, we believe that it would commit to crucial points of physics concerning the geometric phase problems. The generalized coherent state

(GCS) denoted by $|Z\rangle$ is parametrized by a point Z on some manifolds (mostly complex ones)^[10] whose coordinates may be linked to the external or macroscopic parameters characterizing the systems. By considering the propagator for the roundtrip along a closed loop in the generalized phase space, the geometric phase is written as a contour integral of the connection factor that is given as the overlap function between two nearby coherent states. As will be explained in Section 2, the condition determining the geometric phase is given by the variation equation leading to the paths (or orbits) in the parameter space or the generalized phase space. The variation principle plays a role of a substitute for the adiabatic condition determining the adiabatic phases.

In this paper we do not intend to develop a general formalism of the geometric phase, but aim to examine rather specific cases relevant to direct observation by experiment. Namely, we are concerned with the particular cases such that the path is determined by the special class of model Hamiltonians called the resonant Hamiltonian which is given in terms of the linear "generalized spin" in an oscillating "magnetic field".^[11] These model Hamiltonian may be realized in quantum optics or similar device. The generalized spins form the generators of specific Lie groups which are connected with the GCS under consideration. For this particular class of coherent states, the geometric phases are calculated by using the specific solutions called the resonance solutions that are derived by the resonant Hamiltonian. In the following we consider three typical GCS's of compact and non-compact types, namely, the spin (or $SU(2)$) CS, and the Lorentz(or $SU(1,1)$)CS and the boson CS. We show that a detection is possible for the geometric phases accompanying the resonant solutions corresponding to these three types of coherent states.

2. Preliminary

We start with a concise review of the general theory of the geometric phase formulated in terms of the GCS.^[4] Consider the propagator starting from and ending at the state $|Z_0\rangle$ during the time interval T :

$$K = \langle Z_0 | P \exp -\frac{i}{\hbar} \int_0^T \hat{H}(t) dt | Z_0 \rangle \quad (2.1)$$

where the Hamiltonian is time-dependent, which means that the evolution operator generally reads time-ordered product. (2.1) represents the probability amplitude for coincidence; the amplitude for a cyclic change that the system starts with the state $|Z_0\rangle$ and returns to the same state after a time interval T . That implies that the system proceeds along closed paths in the Hilbert space spanned by the set of GCS. Using the partition of unity at each infinitesimal time-interval, we have

$$K = \int \prod_{k=1}^{\infty} \langle Z_{k-1} | Z_k \rangle \exp [-\frac{i}{\hbar} \int_C \langle Z | \hat{H} | Z \rangle dt] D\mu(Z) \quad (2.2)$$

where $D\mu(Z) \equiv \prod_{t=0}^T d\mu(Z(t))$ and $d\mu(Z)$ denotes the invariant measure on the generalized phase space specified by the complex vector $Z = (z_1, z_2, \dots, z_n)$. In (2.2), the infinite product represents the finite connection along the closed loop in the complex parameter space, in which each infinitesimal factor represents the connection between two infinitesimally separated points. If use is made of the approximation $\langle Z_{k-1} | Z_k \rangle \simeq \exp [i \langle Z | \frac{\partial}{\partial t} | Z \rangle dt]$, (2.2) is written as the functional integral over all closed paths

$$K = \int \exp [\frac{i}{\hbar} \Phi(C)] D\mu(Z), \quad (2.3)'$$

where ϕ is nothing the "action functional":

$$\Phi(C) = \int_0^T \langle Z | i\hbar \frac{\partial}{\partial t} - \hat{H}(t) | Z \rangle dt, \quad (2.4)$$

where $H(Z, Z^*, t) \equiv \langle Z | \hat{H} | Z \rangle \equiv H(t)$. Specifically, we write the first term of S as

$$\Gamma(C) = \oint_C \langle Z | i\hbar \frac{\partial}{\partial t} | Z \rangle dt, \quad (2.5)$$

which give nothing but the geometric phase in terms of the GCS, which is quoted simply as Γ hereafter. On the other hand, the second term of (2.4) is called the Hamiltonian term denoted as Δ : $\Delta(C) = \oint_C \langle Z | \hat{H} | Z \rangle dt$. If one uses the kernel function

$$F(Z, Z^*) = \langle \tilde{Z} | \tilde{Z} \rangle$$

($|\tilde{Z}\rangle$ being the unnormalized CS), Γ is cast into the form

$$\Gamma = \oint \sum_{k=1}^n \frac{i\hbar}{2} \left(\frac{\partial \log F}{\partial z_k} dz_k - \frac{\partial \log F}{\partial z_k^*} dz_k^* \right) \equiv \oint \omega \quad (2.6)$$

In order to calculate the explicit form of Γ , we need to select a specific cyclic path $C(Z(t))$ in the generalized phase space. This may be realized by considering the semiclassical limit of (2.4); the stationary phase condition $\delta S = 0$ yielding the equations of motion for Z

$$i\hbar \sum_{j=1}^n g_{ij} \dot{z}_j = \frac{\partial H}{\partial z_i^*}, -i\hbar \sum_{j=1}^n g_{ij} \dot{z}_j^* = \frac{\partial H}{\partial z_i} \quad (2.7)$$

where

$$g_{ij} = \frac{\partial^2 \log F}{\partial z_i \partial z_j^*}, \quad (2.8)$$

which denotes the metric of the generalized phase space: the so-called Kaehler metric. The propagator is thus reduced to a simple form

$$K_{sc} = \exp\left[\frac{i\Gamma(C)}{\hbar}\right] \exp\left[-\frac{i\Delta}{\hbar}\right] \quad (2.9)$$

Namely, if there exist closed paths, the propagator may be expressed as the overlap

between two coherent states

$$K_{sc} = \langle Z_0(T) | Z_0(0) \rangle \quad (2.10)$$

where the ket vector is parametrized by the orbit, the end point of which coincides with Z_0 at the time T. In this way the final state may accomodate the history which the system develops.

To consider the semiclassical limit should be compared with the procedure adopted in getting the adiabatic quantum phase, where the change of state vector is governed by the cyclic motion that evolves adiabatically.^[8] In the present case, the adiabaticity is not necessary and the principle governing the geometric phase is played by the "quantum variational principle" leading to the equation of motion in the parameter space that defines the generalized coherent state. Some explanation is in order regarding the meaning of the choice of the closed path. Taking the semiclassical limit means that the condition is fixed for choosing the specific path among all the possible paths, where the parameter controlling the semiclassical limit is played by the Planck constant. This feature partially corresponds to the situation of taking the adiabatic limit for the quantum transition for which the transition takes place between the states labelled by the discrete eigenstates possessing with the same quantum number.

Here a comment is given for a possible experimental detection of the phase Γ . As is suggested in the above, after a cyclic motion the state vector totally aquires the phase, which by eq. (2.4) consists of two parts: $\Phi = \frac{(\Gamma - \Delta)}{\hbar}$: the geometric and the Hamiltonian term. In this point it is crucial to separate these two terms from each other in actual situation; especially, to extract the geometric term Γ . In this connection, the problem is the fact that it is not easy in general to obtain the phase in a concise manner. The reason arises from the difficulty of finding out the cyclic path relevant to evaluate the phase $\Gamma(C)$. So we must resort to the special situation that enables us to extract the cyclic path in a simple way. In the next section we shall realize this program.

3. The Geometric Phases for Three Types of Coherent States

In this section we consider the three typical coherent states to demonstrate the specific feature of the geometric phase Γ .

3.1. THE CASE OF SU(2) COHERENT STATE

As a first example we examine the case of a spin system which is described by the SU(2) coherent state. Consider a particle with spin \mathbf{J} in a time-dependent magnetic field, which has the component:

$$\mathbf{B}(t) = (B_0 \cos \omega t, B_0 \sin \omega t, B) \quad (3.1)$$

namely, a static field along the z -axis plus a time-dependent field rotating perpendicular to it with the frequency ω , which is familiar in magnetic resonance. The system may be described by the spin or SU(2)CS (alias Bloch state): $|z\rangle$ is defined as

$$|z\rangle = (1 + |z|^2)^{-J} e^{z\hat{J}_+} |0\rangle, \quad (3.2)$$

where $|0\rangle = |J, -J\rangle$ satisfying $\hat{J}_z |0\rangle = -J |0\rangle$ and \hat{J}_\pm are usual spin operators and z takes any complex values. The Hamiltonian of the system is

$$\hat{H}(t) = -\mu \mathbf{B}(\mathbf{t}) \bullet \mathbf{J}, \quad (3.3)$$

where $\mathbf{J} \equiv (\hat{J}_1, \hat{J}_2, \hat{J}_3)$ is a matrix vector satisfying $\mathbf{J} \times \mathbf{J} = i\mathbf{J}$. Note that the state is specified by a single complex parameter expressed in terms of the polar coordinate

$$z = \tan \frac{\theta}{2} e^{-i\phi} \quad (0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi) \quad (3.4)$$

This suggests that the generalized phase space is isomorphic to S^2 in the case of the spin CS . The expectation value of the Hamiltonian $H(t) = H(z, z^*, t)$ is thus

given as

$$H(z, z^*) = H(\theta, \phi) = -\mu J[B_0 \sin \theta \cos(\phi - \omega t) - B \cos \theta] \quad (3.5)$$

Now let us write the variation equation for the case of spin CS. By using the kernel function $F = \langle \tilde{z} | \tilde{z} \rangle = (1 + |z|^2)^{2J}$, we get

$$2iJ\hbar \frac{\partial z}{\partial t} = (1 + |z|^2)^2 \frac{\partial H}{\partial z^*}, \quad (3.6)$$

together with its complex conjugate. This is alternatively written in terms of the polar coordinate,

$$\dot{\theta} = -\frac{1}{J\hbar \sin \theta} \frac{\partial H}{\partial \phi}, \dot{\phi} = \frac{1}{J\hbar \sin \theta} \frac{\partial H}{\partial \theta}. \quad (3.7)$$

For the case of (3.5), it turns out to be

$$\dot{\theta} = -\frac{\mu B_0}{\hbar} \sin(\phi - \omega t), \dot{\phi} = -\frac{\mu}{\hbar} [B_0 \cot \theta \cos(\phi - \omega t) + B], \quad (3.8)$$

One sees that this form of equations of motion allows a special solution

$$\phi = \omega t, \theta = \theta_0 (= \text{const}), \quad (3.9)$$

where the following relation should hold among the parameters θ_0, B, B_0 :

$$\cot \theta_0 = -\left(\frac{B}{B_0} + \frac{\hbar \omega}{\mu B_0} \right). \quad (3.10)$$

The solution of the form (3.9) may be called the "resonance" solution, since it corresponds to the one for the case of the forced oscillation. The set of parameters (B, B_0, ω) satisfying (3.10) for a fixed value $\theta = \theta_0$ belong to a family of resonance solutions. Indeed, this set of parameters forms a surface in the parameter space (B, B_0, ω) , which we call "invariant surface" hereafter and characterizes the resonance condition. Equation (3.9) gives a definite cyclic trajectory (θ, ϕ) with the period $T = 2\pi/\omega$ in the generalized phase space. The condition (3.10) is crucial, since the quantities in the right hand side of (3.10) are all given in terms of constants that may be allowed to be compared with experiment.

Next we turn to the evaluation of the phase Γ that is fitted with this special solution. From (3.2) one gets^[8]

$$\begin{aligned}\Gamma(C) &= \oint \left\langle z \left| i\hbar \frac{\partial}{\partial t} \right| z \right\rangle dt \\ &= \oint \frac{iJ\hbar}{1 + |z|^2} (z^* \dot{z} - c.c.) dt \\ &= \oint J\hbar(1 - \cos \theta) d\phi\end{aligned}\tag{3.11}$$

By noting the resonance solution, this becomes

$$\Gamma(C) = 2\pi J\hbar(1 - \cos \theta_0) = -J\hbar\Omega(C),\tag{3.12}$$

$\Omega(C)$ is nothing but the solid angle subtended by the curve C at the origin of the phase space, which was first used for the case of the adiabatic phase. On the other hand, the Hamiltonian phase Δ is given by

$$\Delta(C) = \frac{2\pi\mu J}{\omega} (B_0 \sin \theta_0 - B \cos \theta_0).\tag{3.13}$$

The important point is that the phase Γ depends only on θ_0 . Therefore any point lying on the "invariant surface" gives the same Γ . This fact may play a crucial role for extracting the geometric part Γ from the total phase that can be detected in possible experimental situations. The detail will be discussed in the next section. On the other hand, the Hamiltonian phase is not determined solely by θ .

3.2. THE CASE OF LORENTZ COHERENT STATE

In this subsection we consider the phase Γ that is connected with a non-compact coherent state; the $SU(1,1)$ CS; (or alternatively called the Lorentz coherent state), since $SU(1,1)$ is locally isomorphic to Lorentz group of 2+1 dimension. First we give a brief explanation for the generators of Lorentz CS. The algebra we need here

is given by a set of bilinear forms of boson creation and annihilation operators for a single mode electromagnetic field:

$$\hat{K}_+ = \frac{1}{2}(\hat{a}^\dagger)^2, \quad \hat{K}_- = \frac{1}{2}\hat{a}^2, \quad \hat{K}_0 = \frac{1}{4}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}). \quad (3.14)$$

It is known that the discrete series of the irreducible representation of $SU(1,1)$ is divided into two classes characterized by the number $k = \frac{1}{4}$ or $k = \frac{3}{4}$. We see that for these two cases $|0\rangle$ represents the state of the photon number zero (namely, the vacuum state) and the state of photon number one respectively. Using this realization, we have the so-called "squeezing operator"

$$S(\zeta) = e^{\zeta\hat{K}_+ - \zeta^*\hat{K}_-} = e^{\frac{1}{2}(\zeta(\hat{a}^\dagger)^2 - \zeta^*\hat{a}^2)} \quad (3.15)$$

with a squeezing parameter $\tanh|\zeta|$ and a rotating angle $\phi/2$.^[12] By applying $S(\zeta)$ to the vacuum state, $|0\rangle \equiv |k, m=0\rangle$, we have the Lorentz CS:

$$|z\rangle = e^{\zeta\hat{K}_+ - \zeta^*\hat{K}_-} |0\rangle = (1 - |z|^2)^k e^{z\hat{K}_+} |0\rangle, \quad (3.16)$$

By using this form, the phase Γ for the Lorentz CS is calculated by following a manner similar to the case of $SU(2)$ CS. In terms of the complex representation, it is given by

$$\Gamma(C) = -i\hbar k \oint_C \frac{z\dot{z}^* - \dot{z}z^*}{1 - |z|^2} dt. \quad (3.17)$$

or using the angle parameters

$$z = \tanh\left(\frac{\tau}{2}\right) e^{-i\phi}, \quad (3.18)$$

we have

$$\Gamma(C) = \int \hbar k (\cosh \tau - 1) \dot{\phi} dt. \quad (3.19)$$

The Lagrangian is given by

$$L(\tau, \phi) = \hbar k (\cosh \tau - 1) \dot{\phi} - H(\tau, \phi) \quad (3.20)$$

and hence the variation equation leads to

$$\dot{\phi} = \frac{1}{\hbar k \sinh \tau} \frac{\partial H}{\partial \tau}, \dot{\tau} = -\frac{1}{\hbar k \sinh \tau} \frac{\partial H}{\partial \phi}. \quad (3.21)$$

Now, consider the system which is composed of cavity mode and the squeezed state generating interaction.^[13] The Hamiltonian is given by

$$\hat{H} = \hbar \omega_0 (\hat{a}^\dagger \hat{a} + \frac{1}{2}) + \hbar [V^* (\hat{a}^\dagger)^2 + V \hat{a}^2], \quad (3.22)$$

where V is the interaction parameter including the effect of pumping light. Here we take $V = \kappa e^{-i\omega t}$: an oscillating pumping light. In terms of the pseudo-spin, the Hamiltonian can be expressed as

$$\begin{aligned} \hat{H} &= 2\hbar [\omega_0 \hat{K}_0 + \kappa (e^{i\omega t} \hat{K}_+ + e^{-i\omega t} \hat{K}_-)] \\ &\equiv \mathbf{C} \bullet \mathbf{K}, \end{aligned} \quad (3.23)$$

where $\mathbf{C} \bullet \mathbf{K} \equiv C_0 \hat{K}_0 - C_1 \hat{K}_1 - C_2 \hat{K}_2$. Here the magnetic field analogue ("pseudo-magnetic field", say) is given by

$$\mathbf{C} = (C_0, C_1, C_2) = 2\hbar (\omega_0, -2\kappa \cos \omega t, -2\kappa \sin \omega t), \quad (3.24)$$

The expectation value of H becomes

$$H(t) \equiv \langle z | \hat{H} | z \rangle = H(\tau, \theta) = 2\hbar k [\omega_0 \cosh \tau + 2\kappa \sinh \tau \cos(\phi - \omega t)]. \quad (3.25)$$

In this way, we get the equation of motion in terms of the angle variables:

$$\dot{\phi} = 2[\omega_0 + \kappa \coth \tau \cos(\phi - \omega t)], \quad \dot{\tau} = 4\kappa \sin(\phi - \omega t), \quad (3.26)$$

We obtain a "resonance solution" in a completely similar manner to the previous

SU(2) case;

$$\phi = \omega t, \tau = \tau_0 (= const), \quad (3.27)$$

provided the following relation is satisfied:

$$\coth \tau_0 = \frac{-\omega - 2\omega_0}{4\kappa}, \quad (3.28)$$

where $(\omega, \omega_0, \kappa)$ should obey the condition

$$|\frac{\omega + 2\omega_0}{4\kappa}| > 1,$$

since $|\coth x| > 1$. Note that the orbit given by (3.27) forms a circle on the "pseudo-sphere". The condition (3.28) just determines the "invariant surface" in the parameter space $(\omega, \omega_0, \kappa)$ on which τ_0 is constant. For the path C described by this solution, the phase Γ becomes a simple form

$$\Gamma(C) = 2\pi\hbar k(\cosh \tau_0 - 1) \quad (3.29)$$

and the Hamiltonian phase is given by

$$\Delta(C) = \frac{4\pi\hbar k}{\omega}(\omega_0 \cosh \tau_0 + \kappa \sinh \tau_0). \quad (3.30)$$

We arrive at a result that is completely parallel with the case of spin CS: the geometric phase depends only on the "invariant surface" and the Hamiltonian phase does not satisfy this property.

3.3. THE CASE OF BOSON COHERENT STATE

As a third example, we consider a harmonic oscillator driven by an external force for which the Hamiltonian is given by

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \omega_0^2 \hat{q}^2) + F(t)\hat{q}, \quad (3.31)$$

This can be written by boson creation and annihilation operators as

$$\hat{H} = \hbar\omega_0\hat{a}^\dagger\hat{a} + \beta(t)\hat{a}^\dagger + \beta^*(t)\hat{a}. \quad (3.32)$$

This type of Hamiltonians appears in the problems of detecting gravitational radiation and/or quantum optics.^[14,15] In the following we shall take up the second one, for example, and discuss the possibility of finding the effect of the geometric phase. Consider a single mode electric field inside a cavity driven externally by a coherent driving field. If we neglect the cavity damping, we have the Hamiltonian:

$$\hat{H} = \hbar\omega_0\hat{a}^\dagger\hat{a} + \hbar(\hat{a}^\dagger E(t)e^{-i\omega t} + \hat{a}E^*(t)e^{i\omega t}), \quad (3.33)$$

which belongs to the type of (3.32). The first term represents the cavity mode Hamiltonian, where ω_0 means the fundamental cavity resonance and the second term gives the Hamiltonian for the coherent driving field respectively. Here $E(t)$ is the driving field amplitude, while ω means the driving frequency. The coherent state is now given by a standard (boson) coherent state:

$$|z\rangle = e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^\dagger} |0\rangle \quad (3.34)$$

where the relation $\hat{a}|z\rangle = z|z\rangle$ holds; z is proportional to complex amplitude of the classical electromagnetic field obtained as the solution of Maxwell equation.^[15]

The variation equation now becomes

$$\dot{z} + i\omega_0 z = -iEe^{-i\omega t}, \quad (3.35)$$

or, using the polar form $z = re^{i\theta}$,

$$\dot{r} = -E \sin(\theta + \omega t), \quad r(\dot{\theta} + \omega_0) = -E \cos(\theta + \omega t). \quad (3.36)$$

In a similar manner to the previous sections, one can also have a “resonance solution” with period $T = \frac{2\pi}{\omega}$ which is given as

$$r = r_0, \quad \theta = -\omega t, \quad (3.37)$$

where the relation

$$r_0 = \left| \frac{E}{\omega + \omega_0} \right| \quad (3.38)$$

defines the “invariant surface”. The phase Γ is thus evaluated as

$$\Gamma(C) = 2\pi\hbar r_0^2. \quad (3.39)$$

On the other hand, the phase Δ becomes

$$\Delta(C) = \frac{2\pi}{\omega}(\hbar\omega_0 r_0^2 - 2Er_0) \quad (3.40)$$

As in the previous two cases, the phase Γ depends only on the invariant surfaces and the Hamiltonian phase does not possess with such a property.

4. Possible Detection of the Geometric Phases

We shall examine possible experimental detection of the geometric phases. We first consider a general setting to detect the phase with the aid of interference by using the "particle beam" by which the coherent state is conveyed.

We suppose an interference apparatus consisting of two "arms". (see Fig.1). Consider the incident beam in which the coherent state is initially in the $|z\rangle$ and it is splitted into two beams running along two "arms". At the initial junction point (the point A in the Fig.1), the state is set to be in the state $|z\rangle$. The interference can occur in the following manner: The state in one beam is set to be in the same state as the initial one $|z\rangle$, whereas the state in the other beam is arranged such that the magnetic field or pseudo-magnetic field is applied on this beam; hence, after the time interval T , it becomes $U(T)|z\rangle$. Thus if one considers the recombination at the final junction point (the point B in the Fig.1), the interference may be given by the superposition:

$$|\psi\rangle = \frac{1}{2}(|z\rangle + U(T)|z\rangle)$$

Actually, the interference can be observed by the overlap $\langle\psi|\psi\rangle$, which turns out to be

$$\langle\psi|\psi\rangle = \frac{1}{4}(2 + \langle z|U(T)|z\rangle + \langle z|U^\dagger(T)|z\rangle)$$

The cross term gives nothing but the propagator for a round trip from z to z , hence

$$\langle\psi|\psi\rangle = \frac{1}{2}(1 + \cos\Phi(C)) = \cos^2\frac{\Phi(C)}{2}$$

Here we have two problems: the first is the problem of "coincidence", namely, the time interval T should match the frequency ω appearing in the oscillatory magnetic or pseudo-magnetic field. This suggests that during the travel of the beam along the one arm whose length is taken to be $L = cT$, (c means the beam velocity) the spin or pseudo-spin figures the closed loop in the Bloch sphere or pseudo-sphere in the complex z space. Having assumed that this condition is satisfied, we expect the effect of the interference due to the phase Γ .

The second point to be mentioned is the problem concerning how one can separate the geometric part Γ from the total phase Φ . This may be possible, if one takes account of the characteristics of the resonance solutions; namely, if we choose the parameters (B, B_0, ω) or $(\kappa, \omega, \omega_0)$ such that the dynamical term Δ vanishes, we can extract the effect which comes only from the geometric part Γ alone. On the basis of this general argument, we examine the condition for which the phase Γ vanishes. In order to see this, we consider the three different cases separately.

(1): For the case of SU(2) CS, the condition for which Δ vanishes reads

$$\cot \theta_0 = \frac{B_0}{B}, \quad (4.1)$$

By combining this with the relation of "invariant surface" (3.10), we get

$$\omega = -\frac{\mu(B_0^2 + B^2)}{\hbar B}. \quad (4.2)$$

The phase Γ thus becomes

$$\Gamma(C) = 2J\pi \left(1 - \frac{B_0}{\sqrt{B_0^2 + B^2}}\right). \quad (4.3)$$

As to the actual setting of experiment, this may be realized by the particle beam consisting of particles of spin J . One of the splitted two beams is prescribed to be placed in the magnetic field that oscillates sinusoidally.

(2): For the case of the Lorentz coherent state, the condition for which the dynamical phase vanishes is given by

$$\coth \tau_0 = -\frac{\kappa}{\omega_0}, \quad (4.4)$$

where the inequality $|\frac{\kappa}{\omega_0}| > 1$ should be satisfied. If we combine this with the

relation between the angle θ_0 and (κ, τ, ω) , we get

$$4\kappa^2 = (2\omega_0 + \omega)\omega_0.$$

The phase Γ is calculated to be

$$\Gamma(C) = 2k\pi\hbar\left(\frac{|\kappa|}{\sqrt{\kappa^2 - \omega_0^2}} - 1\right). \quad (4.5)$$

In this case, the particle beam can be taken as the coherent light (laser) beam; the coherent state is realized by the squeezed state, which may be prepared appropriately. If one of two splitted beams is controlled by pumping which oscillates sinusoidally, we can expect the interference pattern due to the geometric phase according to the general formula.

(3): In a very similar manner, we can also arrange the experiment for the case of the canonical coherent state, for which the condition for $\Delta(C)$ to vanish is given by

$$\omega = -\frac{1}{2}\omega_0. \quad (4.6)$$

The phase Γ is given by

$$\Gamma(C) = \frac{8\pi}{\omega^2}|E|^2. \quad (4.7)$$

In this case, the experimental demonstration may also be carried out by using the laser beam.

5. Discussion and Summary

As is seen from the result of the previous section, the geometric phases nicely match the change of external field; typically, sinusoidal oscillation characterizing the resonant Hamiltonian. For this case, there exists a simple path on the generalized phase space of CS (which is called the “resonance solution”) when the external parameters satisfy a certain condition(“invariant surface”). In this way, the geometric phase depends only on the “invariant surface”. This feature enables us to arrange the experiment such that the effect of the phase is discriminated from the dynamical(or Hamiltonian) phase which depends on the explicit form of the Hamiltonian.

From the point of view of differential geometry, the appearance of the geometric phases can be regarded as a manifestation of the ”holonomy” in quantum mechanics.^[7] As we have seen, the appearance of the geometric phase is relevant only for the case of non-stationary problem, namely, the time-dependent Hamiltonian. Indeed this is very contrast to the situation of the time-independent Hamiltonian. Here a remark is given for this point. First to be mentioned is that in the case of stationary case, the quantity with which we are primarily interested is the energy eigenstate. Thus what is a connection between the energy eigenstate and the phase Γ ? Suppose an isolated system that is placed in the constant external field. Then the expectation value of the Hamiltonian is time-independent and the motion of the parameter Z that determines the phase Γ lies on the surface of constant energy: $H(Z, Z^*) = E$. For this case, after a cyclic change the semi-classical transition amplitude K_{sc} acquires the phase factor except for the energy factor: $\exp[i\Phi] = \exp[\frac{\Gamma(C)}{\hbar}]$ For an isolated system K_{sc} should be single-valued with respect to Z ; namely,

$$\exp[i\Gamma(C)] = 1 \quad (5.1)$$

This is a reminiscent of the singlevalued nature of the usual wave function leading

to Bohr-Sommerfeld quantization, thus^[16]

$$\Gamma(C) = \oint_C \langle Z | i\hbar \frac{\partial}{\partial t} | Z \rangle dt = 2\pi\hbar n(n : \text{integer}). \quad (5.2)$$

We here demonstrate the above statement by using the simplest model Hamiltonian: $\hat{H} = -\mu B \hat{J}_z$ and spin CS. Then $H(Z, Z^*) = \mu B J \cos \theta = E (= \mu B m)$, since $\cos \theta = \frac{m}{J}$. Therefore

$$\Gamma(C) = 2\pi J \hbar (1 - \cos \theta_0) = 2\pi \hbar (J - m), \quad (5.3)$$

On the other hand, we know m (the z-component of the spin) takes quantized integer (or half integer value) ranging from $-J$ to $+J$. Thus, if Γ is exponentiated, we have the trivial result. In this way, the phase Γ should be called a "non-holonomic phase", if it is used for the stationary state. On the contrary, we have the "holonomical" phase only for the case when we consider the non-stationary quantum state for which the concept of energy eigenstates loses the meaning, that is, we have the concept of quasi-energy at best.

Finally, we shall point out possible perspectives on the utility of the geometric phase inspired from the resonant Hamiltonian that has not been treated here. The Hamiltonian in the external field considered here is generic and there may be possible phenomena that can be described by these simple model Hamiltonians: for example, the Hamiltonian relating to the Bogoliubov equation in the superfluid He³ and similar models inspired from condensed matter physics.

Acknowledgements:

The authors would like to thank the members of Department of physics, Ritsumeikan university during this work was carried out, especially, Mr.M.Muzui for his help in preparing the figure. One of the authors (H.K.) is grateful to professor H.Rauch of the Atom-Institut, Austrian Universities at Vienna for kindly inviting him to deliver a seminar about the subject discussed in the present paper.

APPENDIX A

We summarize some necessary formulas for the Lorentz coherent states.^[10] The discrete series generators $K_i(i = 1, 2, 3)$ of SU(1,1) algebra satisfy the followig commutation relations :

$$[\hat{K}_0, \hat{K}_1] = i\hat{K}_2, \quad [\hat{K}_1, \hat{K}_2] = -i\hat{K}_0, \quad [\hat{K}_2, \hat{K}_0] = i\hat{K}_1, \quad (\text{A.1})$$

which can be written formally as $[\hat{K}_l, \hat{K}_m] = i\tilde{\epsilon}_{lmn}\hat{K}_n$, where the symbol $\tilde{\epsilon}$ is the same as the one appearing in the "pseudo" scalar product (3.23). These are the abbreviavion of the usual commutation relation

$$[\hat{K}_0, \hat{K}_\pm] = \pm\hat{K}_\pm, \quad [\hat{K}_-, \hat{K}_+] = 2\hat{K}_0, \quad (\text{A.2})$$

where $\hat{K}_\pm = i(\hat{K}_1 \pm i\hat{K}_2)$ are raising and lowering operators of a SU(1,1) state. The eigenvectors of K_0 are specified by (k,m):

$$\hat{K}_0 |k, k+m\rangle = (k+m) |k, m\rangle, \quad (\text{A.3})$$

where k is a real number determined by the representation of SU(1,1) algebra and m is a non-negative integer. Specifically, $|0\rangle \equiv |k, m=0\rangle$ becomes a starting state vector from which the Lorentz coherent state is constructed. If we use

$$|m\rangle \equiv |k, k+m\rangle = \left[\frac{\Gamma(2k)}{m!\Gamma(m+2k)} \right]^{1/2} (\hat{K}_+)^m |0\rangle, \quad (\text{A.4})$$

we get the explicit form of the CS

$$|z\rangle = (1 - |z|^2)^k \sum_{m=0}^{\infty} \left[\frac{\Gamma(m+2k)}{m!\Gamma(2k)} \right]^{1/2} z^m |m\rangle. \quad (\text{A.5})$$

Furthermore noting that $|m\rangle$'s are mutually orthonormal, we get the overlap for

the un-normalized CS;

$$\langle z_1 | z_2 \rangle = \frac{[(1 - |z_1|^2)(1 - |z_2|^2)]^k}{(1 - z_1^* z_2)^{2k}}. \quad (\text{A.6})$$

Next some matrix elements are given for the generators of SU(1,1) algebra by following the the same procedure for the case of SU(2) CS.^[9] The matrix elements for \hat{K}_+ is calculated as follows: Differentiating (3.16) with respect to z_2 , we have

$$\frac{d}{dz_2} (1 - |z_2|^2)^{-k} \langle z_1 | z_2 \rangle = \langle z_1 | \frac{d}{dz_2} e^{z_2 \hat{K}_+} | 0 \rangle = (1 - |z_2|^2)^{-k} \langle z_1 | \hat{K}_+ | z_2 \rangle. \quad (\text{A.7})$$

If using (A.6), this leads to

$$\frac{\langle z_1 | \hat{K}_+ | z_2 \rangle}{\langle z_1 | z_2 \rangle} = \frac{2k z_1^*}{1 - z_1^* z_2}. \quad (\text{A.8})$$

In a similar manner, we also get

$$\frac{\langle z_1 | \hat{K}_- | z_2 \rangle}{\langle z_1 | z_2 \rangle} = \frac{2k z_2}{1 - z_1^* z_2}. \quad (\text{A.9})$$

To derive the matrix element of \hat{K}_0 , it is convenient to use use the formula:

$$e^{-z \hat{K}_+} \hat{K}_0 e^{z \hat{K}_+} = \hat{K}_0 + z \hat{K}_+,$$

which yields

$$\langle z_1 | \hat{K}_0 | z_2 \rangle = (1 - |z_2|)^k \langle z_1 | e^{z_2 \hat{K}_+} (\hat{K}_0 + z_2 \hat{K}_+) | 0 \rangle = k \langle z_1 | z_2 \rangle + z_2 \langle z_1 | \hat{K}_+ | z_2 \rangle.$$

By using this together with (A.8), we have

$$\frac{\langle z_1 | \hat{K}_0 | z_2 \rangle}{\langle z_1 | z_2 \rangle} = \frac{k(1 + z_1^* z_2)}{1 - z_1^* z_2}. \quad (\text{A.10})$$

APPENDIX B

We point out a possible connection between the resonance solutions and the NMR (nuclear magnetic resonance). The NMR has been used for the study of the adiabatic phase in a different context (See for example,^[17]). First to be noted is that the semiclassical solution satisfying $\delta S = 0$ gives the exact solution for the CS solution for the Schrödinger equation. Note that the CS does not change its form, since the Hamiltonian (3.3) involves the generators linearly.^[10] Therefore we consider the solution of Schrödinger equation. Let us first remind of the basic point of the NMR briefly. Consider that the time evolution is given by $\hat{U}'(t) = e^{-\frac{i}{\hbar}\hat{H}'t} (\hat{H}' \equiv -\mu\mathbf{B}'' \bullet \hat{\mathbf{J}})$

in the moving frame (F') which rotates in x-y plane with the angular velocity ω . On the other hand, in the static frame, say F , we have $\hat{U}(t) \equiv e^{-\frac{i}{\hbar}\int \hat{H}(t) dt} = e^{-i\omega\hat{J}_z t} \hat{U}'(t)$, where B'' represents the effective magnetic field in the moving frame^[18] which is given by

$$\mathbf{B}'' = \mathbf{B}' + \frac{\hbar}{\mu}\omega = (B_0, 0, B) + (0, 0, \frac{\hbar\omega}{\mu}) = (B_0, 0, B + \frac{\hbar\omega}{\mu}) \quad (\text{B.1})$$

Note that the operators in both frames are written in the Schrödinger picture for each frame. The spin variables in both frames satisfy the equation of motion.

$$\frac{d'}{dt}\mathbf{S} = \frac{d}{dt}\mathbf{S}' = \frac{\mu}{\hbar}\mathbf{S}' \times \mathbf{B}'', \quad (\text{B.2})$$

which shows that \mathbf{S}' makes a precession about the magnetic field \mathbf{B}'' , namely, this means that the spin nutates in the static frame.^[19] If we have $\mathbf{S}'/\!/ \mathbf{B}''$ at $t = 0$, it follows that \mathbf{S}' (\mathbf{S} as seen from F') becomes constant in the moving frame F' ,

$$\mathbf{S}'(t) = \mathbf{S}(t = 0) = (J \sin \theta_0, 0, -J \cos \theta_0), \quad (\text{B.3})$$

where

$$\cot \theta_0 = - \left(\frac{B}{B_0} + \frac{\hbar\omega}{\mu B_0} \right). \quad (\text{B.4})$$

Furthermore, this shows that in the frame F , \mathbf{S} purely rotates with the same

frequency ω as the magnetic field in the x-y plane:

$$\mathbf{S}(t) = (J \sin \theta \cos \omega t, J \sin \theta \sin \omega t, -J \cos \theta)$$

If comparison is made with (B.2), this leads to

$$\phi = \omega t. \quad (\text{B.5})$$

(B.4) and (B.5) yield exactly the same conditions for (θ, ϕ) as (3.9) and (3.10) for "resonance" solution and "invariant surface". In particular, if the condition

$$\omega = \omega_0 \equiv -\frac{\mu B}{\hbar}$$

satisfies, the magnetic resonance occurs. Finally, the similar argument may be applied for the resonant Hamiltonian for the Lorentz coherent states, which can be simply done by replacing the spin and the magnetic field by the "pseudo-spin" and "pseudo-magnetic field".

REFERENCES

1. P. A. M. Dirac, Fields and Quanta **3**(1972)139.
2. A. Shapere and F. Wilczek eds. " Geometric Phases in Physics, World Scientific (1989).
3. P.A.M. Dirac, Proc. Roy. Soc.**A133**(1931)60.
4. H. Kuratsuji, Phys. Rev. Lett.**61**(1988)1687 and references cited therein.
5. G. Hertzberg and H. C. Longuet-Higgins Dics. Farad. Soc. **35**(1963)77; A.J.Stone, Proc.Roy.Soc.**351**(1976)141.
6. M. Berry, Proc.Roy.Soc.**392**(1984)45
7. B.Simon, Phys.Rev.Lett.**51**(1983)2167.

8. H.Kuratsuji and S.Iida, Prog.Theor.Phys, Phys. **74**(1985)439.
9. H. Kuratsuji and T. Suzuki, J. Math. Phys.**21**(1980)472.
10. A. M. Perelomov," Generalized Coherent State", (Springer, Berlin,1985)
11. H.Kuratsuji and M.Matsumoto; Phys.Lett.**A155**(1991)99.
12. See e.g. C. M. Caves, Phys. Rev. **D23** (1981)1693.
13. P. D. Drummond and D. F. Walls, J. Phys. A **13**(1980),725.
14. J. N. Hollenhorst, Phys. Rev. D **19**(1979)1669.
15. R. Loudon, "Quantum Theory of Light" (Pergamon press,
Oxford, 1973)
16. H. Kuratsuji, Phys.Lett. **108B** (1982) 367.
17. J.A.Cina, Chem.Phys.Lett.**132**(1986)393.
18. A. Messiah "Mécanique Quantique" (Dunod, Paris, 1958)
19. N. Bloembergen, "Nuclear Magnetic Relaxation" (W.A.Benjamin, New
York, 1961);I. I. Rabi, N. F. Ramsey and J. Schwinger, Rev. Mod. Phys.**26**
(1954),167